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NEWTON CONTROL LAWS FOR NONLINEAR CONTROLLER DESIGN

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ABSTRACT

Strong similarities between control theory and the theory on the solution of operator equations have been observed and basic results in control theory have been derived from operator theory arguments. The purpose of this work is to use the underlying duality in order to develop analysis and synthesis techniques for nonlinear systems. As an example, controllers induced by the Newton method are introduced and the corresponding stability characteristics are studied. The concepts are demonstrated by applications to linear and nonlinear systems.

INTRODUCTION

Control theory has had positive interactions with operator theory in the past. A number of control researchers have noticed (Aström and Wittenmark[1]) and some have used an underlying duality between control theory and the theory on the solution of operator equations, to establish strong quantitative results. It suffices to mention a few: Kalman[2] was first to use contraction principle arguments to study the stability of autonomous discrete systems. Zames[3] then used the same principle to derive the so called small gain, circle and conicity stability conditions for continuous input-output systems. Much later, the singular value decomposition method, originally introduced in the study of the sensitivity of linear operator inversion, was employed by Doyle and Stein[4] and Lehtomaki[5] to establish a theory on the robustness of linear feedback structures. On the other hand, basic control theory results appear in operator theory. For example, the von Neumann convergence analysis for linear partial differential equation solution is a basic application of the Nyquist stability criterion.

Practically all these results are confined to analysis issues, such as stability and robustness. The implications for synthesis and design have yet to be studied. The focus of this work is feedback controller design: if the controller design problem could be formulated as an operator equation, it could benefit in both the analysis and synthesis aspects from a relatively well developed theory on the solution of operator equations. For linear systems no major gains are to be expected, since the implied operator inversion has been either explicitly (Garcia and Morari[6]) or implicitly (Stein[7]) used in control studies, although some insight in the issue of "inverting control" might be gained. Compared to linear systems however, there are very few results in nonlinear controller design and these are limited to stability analysis (Zames[3], Popov[8]). Only limited attempts towards a general synthesis theory have been reported, the most successful perhaps being the global linearization theory of Hunt, Su and Meyer[9]. Some open questions, seemingly inherent in the method (robustness of the linearization transformation, interpretation of the transformed inputs and outputs), leave open space for a different approach to the problem at hand.

The purpose of this paper is to establish the duality between controller design and algorithm development for the solution of operator equations. A number of meaningful control objectives can be formulated as operator inversion problems, which in turn have good practical as well as theoretical support. This framework allows us to address nonlinear controller design in a general and intuitively clear manner. At the present stage, no hope is expressed to exhaust the subject, but rather to expose a concept and illustrate its applications.

In Section II the notation, some basic notions and necessary computational tools are introduced. In Section III a framework for the stability analysis of discrete open and closed loop nonlinear systems is developed, based on the Contraction Mapping Principle. Control law synthesis is discussed in Section IV and control laws are derived based on the Newton method for linear and nonlinear systems. Examples of the control law applications to linear and nonlinear systems are included in Section V. Section VI summarizes and concludes the paper.

II. PRELIMINARIES

Assumptions:

The systems considered are governed by the vector ordinary differential equations:

$$\frac{\partial x}{\partial t} = f(x,u(t)) \tag{1}$$

where $x \in R^n$ is the state of the system and for every $t \in [0,\infty)$, $u(t) \in R^m$ is the input, with the corresponding output map ($y \in R^m$):

$$y = g(x) \tag{2}$$

Existence and uniqueness of solutions of (1) are assumed. In the present stage of research, exact modelling is assumed and the state vector is fully accessible. For open loop stable systems, the assumption on the accessibility of the states is not restrictive since any unknown state component will dissipate with time. It is, however, crucial for unstable systems. The system inputs will be assumed to be piecewise constant functions to reduce the problem at hand to a finite dimensional space.

Notation: The letter s is used as a superscript to mark the discrete time. The s^{th} sampling interval extends from t^s to t^{s+1}. $T = t^{s+1} - t^s$ is the (constant) sampling time; x^s is the state at t^s; u^s is the system input, held constant over $(t^s, t^{s+1}]$.

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In the discrete setting of the study, $\chi(t_2;t_1,x,u)$ is the solution of (1) at time t_2 for $u(t) = u(t_1 < t \leq t_2)$, and initial condition $\chi(t_1;t_1,x,u) = x$; χ^s will denote the state of the system at $t = t^{s+1}$, i.e. x^{s+1}:

$$\chi^s \overset{\text{def}}{=} x^{s+1} = \chi(t^s+T;t^s,x^s,u^s) \tag{3}$$

Since (1) is stationary: $\chi(t_1+\Delta t;t_1,x,u) = \chi(t_2+\Delta t;t_2,x,u)$. Therefore time will be dropped from the parameter list and the following convention will be used:

$$\chi^s = \chi(T;x^s,u^s) = \chi(t^s+T;t^s,x^s u^s) \tag{4}$$

The derivatives of χ^s with respect to x^s and u^s will be $\Phi^s \left[\overset{\text{def}}{=} \frac{\partial \chi^s}{\partial x^s}\right]$ and $\Gamma^s \left[\overset{\text{def}}{=} \frac{\partial \chi^s}{\partial u^s}\right]$ respectively. y^s $[\overset{\text{def}}{=} g(x^s)]$ is the system output at t^s. Finally $C(s+1)\left[\overset{\text{def}}{=} \frac{\partial g(x^{s+1})}{\partial x^{s+1}}\right]$ is the derivative of the output map with respect to the state at $s+1$.

State Derivatives

The state derivatives with respect to initial conditions (Φ^s) and inputs (Γ^s) frequently appear throughout the paper. In the following a computational theory for these and related quantities is presented. The statements are proved in Economou[10]. Φ^s is the solution at $t = t^{s+1}$ of the initial value problem:

$$\frac{\partial \Phi(t)}{\partial t} = \frac{\partial f(\zeta,\xi)}{\partial \zeta}\bigg|_{\substack{\zeta=\chi(t;x^s,u^s)\\ \xi=u^s}} \cdot \Phi(t) \tag{5}$$

with initial conditions

$$\Phi(t^s) = I \tag{6}$$

For a linear system (5) and (6) can be integrated explicitly, yielding;

$$\Phi^s = \exp(AT) \tag{7}$$

where A is the state feedback matrix of the state space realization of the system.

Γ^s is the solution at $t = t^{s+1}$ of the initial value problem

$$\frac{\partial \Gamma(t)}{\partial t} = \frac{\partial f(\zeta,\xi)}{\partial \zeta}\bigg|_{\substack{\zeta=\chi(t;x^s,u^s)\\ \xi=u^s}} \cdot \Gamma(t) + \frac{\partial f(\zeta,\xi)}{\partial \xi}\bigg|_{\substack{\zeta=\chi(t;x^s,u^s)\\ \xi=u^s}} \tag{8}$$

with initial conditions

$$\Gamma(t^s) = 0 \tag{9}$$

For linear systems:

$$\Gamma^s = [\exp(AT)-I]A^{-1}B \tag{10}$$

where B now is the input matrix of the state space representation of the system.

In a similar manner the second and higher order derivatives can be computed.

System Operator Under the existence and uniqueness assumptions, systems governed by (1) generate a well defined operator, which maps states x^s at the beginning of a sampling interval t^s and inputs u^s constant over that sampling interval, to states $x^{s+1} = \chi(T;x^s,u^s)$ and outputs $y^{s+1} = g(x^{s+1})$ at t^{s+1}. The system operator is denoted by N

$$R^n \times R^m \ni (x,u) \overset{N}{\rightarrow} (\chi,y) \in R^n \times R^m \tag{11}$$

Control Objectives. The basic control objective used to formulate the control problem as an operator equation solution problem, is to drive the system to a steady state $(x^{s+1}=x^s)$ with its output y^{s+1} at a desired level y^*, i.e., $y^{s+1} = y^*$.

An alternative (and as it turns out, simpler) definition of the control objective, is to disregard the state evolution and opt for driving the system output at $y^{s+1} = y^*$, or, in a more general fashion at $y^{s+n} = y^*$, where n is a fixed number of forward steps.

Control Operator Equations. To each control objective corresponds an operator equation. The following operator equations are generated by the objectives above respectively

$$N(x,u) = (x,y^*) \tag{12}$$

and (for n=1)

$$[0 \quad I_m]\, N(x,u) \overset{\text{def}}{=} N_1(x,u) = y^* \tag{13}$$

I_m being the identity matrix.

Control law computations to achieve the objective can be based on iterative algorithms for the solution of (12) and (13). Potential gains of this approach stem from a well developed theory on algorithms for the solution of operator equations, especially in areas where control theory has not progressed as much, as is the case of nonlinear systems.

A number of important issues in controller design such as stability, performance, robustness, etc., have their well studied counterparts in the theory of operator equations: convergence, speed of convergence, sensitivity to approximation error, etc. and we will try to take advantage of this duality.

III. STABILITY ANALYSIS VIA THE CONTRACTION MAPPING PRINCIPLE

To study the stability of the control laws that arise from iterative operator equation solution algorithms, analysis methods are developed in this section. The Contraction Mapping Principle (CMP) is best suited to the problem at hand and serves as the basis for the analysis. At every step the implications for linear systems are studied. Finally the relation with established techniques, as is the case of the indirect Lyapunov method, are discussed.

Control laws derived from operator equation arguments will be shown to be of the form

$$u^{s+1} = \psi(x^s,u^s) \tag{14}$$

where ψ is some finite dimensional operator from $R^{n\times m}$ to R^m. In order to study the stability of related control laws, first some basic corollaries of the CMP are stated. Stability conditions for discrete open loop systems generated by sampling systems of the form (1)

$$x^{s+1} = \chi(T;x^s,u^s) \tag{15}$$

are derived. These results are then extended for closed loop stability analysis of discrete systems, because (15) augmented by (14) constitutes an open loop system for the augmented "state" vector $\begin{bmatrix} x^s \\ u^s \end{bmatrix}$. The statements and theorem are proved in Economou[10]. The results hold for any vector norm and its associated induced operator norm.

<u>Contraction Principle Lemma</u>. Assume that the operator F on a Banach space is differentiable in a ball $U(x^0,r)$ of center x^0 and radius r, where

$r \geq r^0 \stackrel{\text{def}}{=} ||F(x^0) - x^0||/(1-\theta)$ and that

$$||F'(x)|| \leq \theta < 1, \quad x \quad U(x^0,r) \tag{16}$$

Then the sequence

$$x^{s+1} = F(x^s),\ s = 1,2,\ldots,\ x^1 = \bar{x} \tag{17}$$

converges to the unique solution x^* of the operator equation

$$x^* = F(x^*) \tag{18}$$

in $U(x^0,r)$ for every $\bar{x} \subset U(x^0,r^0)$.

<u>Open Loop Stability</u>

<u>Definition 1</u>. Consider the discrete system (15) generated by (1) and assume a fixed input to the system u_f, (u_f R^m). x_{eq} is called an <u>equilibrium state</u> of the discrete system if

$$x_{eq} = \chi(T;x_{eq},u_f) \tag{19}$$

<u>Definition 2</u>. A ball $U(x^0,r)$ is called a <u>region of attraction</u> for the equilibrium point x_{eq} of the discrete system generated by (1) and $u = u_f$, if every trajectory starting at any initial state within $U(x^0,r)$ eventually converges to x_{eq}.

<u>Theorem 1</u>. Consider the discrete open loop system generated by (1) with $u = u_f$

$$x^{s+1} = \chi(T;x^s,u_f) \tag{20}$$

and a state x^0. If

$$||\Phi(x,u_f)|| \stackrel{\text{def}}{=} \left|\left| \frac{\partial\chi(T;x,u_f)}{\partial x} \right|\right| \leq \theta < 1,\ \forall x \in U(x^0,r) \tag{21}$$

where $r \geq r^0 \stackrel{\text{def}}{=} ||\chi(T;x^0,u_f) - x^0||/(1-\theta)$, then the system has a unique asymptotically stable equilibrium state x_{eq} in $U(x^0,r)$. Furthermore, $U(x^0,r^0)$ is a region of attraction for x_{eq}.

<u>Corollary 1</u>. A discrete linear system is (globally) asymptotically stable if and only if

$$\rho(\Phi) = \rho(\exp(AT)) < 1 \tag{22}$$

with ρ denoting spectral radius.

Corollary 1 merely states that a discrete linear open loop system is stable if and only if the eigenvalues of the state feedback matrix Φ are inside the unit circle.

<u>Closed Loop Stability</u>

Consider the discrete closed loop system consisting of the open loop system (15) and feedback control law (14). Augmenting (15) by (14) generates an open loop system for the augmented state vector $\begin{bmatrix} x^s \\ u^s \end{bmatrix}$. The stability of the closed loop system is equivalent to the stability of the augmented open loop system and is characterized by the following theorem.

<u>Theorem 2</u>. Consider the discrete closed loop system generated by augmenting a sampled system of the form (15) with a feedback control law of the form (14) as well as a state $\begin{bmatrix} x^0 \\ u^0 \end{bmatrix}$ of the resulting system. If

$$\left|\left| \begin{matrix} \dfrac{\partial\chi(T;x,u)}{\partial x} & \dfrac{\partial\chi(T,x,u)}{\partial u} \\ \dfrac{\partial\psi(x,u)}{\partial x} & \dfrac{\partial\psi(x,u)}{\partial u} \end{matrix} \right|\right| \leq \theta < 1,$$

$$\forall (x,u) \in U((x^0,u^0),r) \tag{23}$$

where $r \geq r^0 \stackrel{\text{def}}{=} ||\chi(T;x^0,u^0) - x^0, \psi(x^0,u^0)-u^0)||/(1-\theta)$, the closed loop system has a unique asymptotically stable equilibrium state (x_{eq}, u_{eq})in $U((x^0,u^0),r)$. Furthermore $U((x^0,u^0),r)$ is a region for attraction for (x_{eq},u_{eq}).

<u>Corollary 2</u>. A linear system with a linear control law

$$u^{s+1} = \Psi x^s + \Omega u^s,\quad \Psi \in R^{mxn},\quad \Omega \in R^{mxm} \tag{24}$$

is stable if and only if

$$\rho\begin{bmatrix} e^{AT} & (e^{AT}-I)A^{-1}B \\ \Psi & \Omega \end{bmatrix} < 1 \tag{25}$$

Corollary 2 implies that the closed loop system will be stable, if and only if the feedback law (24) places the closed loop poles of the discrete system inside the unit circle.

<u>Remarks</u>

It is interesting to discuss how Theorem 1 relates to the Lyapunov stability of the nonlinear system (1): Suppose x_{eq} is an equilibrium point of (1). If (21) holds for all $x \in U(x^0,r)$ it is also true for x_{eq}. That is

$$||\Phi(x_{eq},u_f)|| < 1 \tag{26}$$

However, for $x = x_{eq}$, the solution to (1) is: $\chi(t;x_{eq},u_f) = x_{eq}$. Then (5) becomes

$$\frac{\partial\Phi(t)}{\partial t} = \partial f(\zeta,\xi)/\partial\zeta \Big|_{\substack{\zeta=x_{eq} \\ \xi=u_f}} \Phi(t) = A_{eq}\Phi(t) \tag{27}$$

with A_{eq} a constant matrix, the state feedback matrix of the local linearization of (1) at x_{eq}. The unique solution to (27) at $t = t + T$ is

$$\Phi = \exp(A_{eq}T)$$

It is known that for any induced matrix norm, $\rho(M) < ||M||$ (Desoer and Vidyasagar[11]), therefore (26) and (27) imply that

$$\rho(\exp(A_{eq}T)) < 1$$

and subsequently the linearized system at x_{eq} is stable. It follows that the equilibrium state of the nonlinear system is stable in the sense of Lyapunov if it is stable in the sense of Theorem 1.

The main feature of Theorem 1 is that it is not confined to local stability analysis (infinitesimal perturbations), but it establishes the stability of the nonlinear system to finite perturbations and yields a region of attraction for the equilibrium point.

IV. CONTROL LAW SYNTHESIS VIA THE NEWTON METHOD

Contraction Principle and Hybrid Newton Control Laws were introduced in Economou[10] to solve the control operator equation (12). In the following, some additional control laws for the solution of (13) are developed. To simplify the notions involved, only the case $n = 1$ (where n is the number of forward steps allowed to achieve the desired output y^*) is treated. To distinguish from the Hybrid Newton Control Laws, the resulting controllers are called quasi-Newton, the nomenclature to be justified in the following.

1. First Quasi-Newton Control Law

In order to solve (13), expand N_1

$$N_1(x,u) = y = g(\chi(T;x,u)) \tag{28}$$

in its Taylor series around a state $x = x^s$, $u = u^s$ and subtract y^*:

$$\begin{aligned} g(\chi(T;x^{s+1},u^{s+1})) - y^* &= g(\chi(T;x^s,u^s)) - y^* \\ &+ \partial g(\zeta)/\partial\zeta\Big|_{\zeta=\chi(T;x^s,u^s)} \frac{\partial\chi(T;x^s,u^s)}{\partial x^s}(x^{s+1}-x^s) \\ &+ \partial g(\zeta)/\partial\zeta\Big|_{\zeta=\chi(T;x^s,u^s)} \frac{\partial\chi(T;x^s,u^s)}{\partial u^s}(u^{s+1}-u^s) \\ &+ 0\left[\left|\left| \begin{matrix} x^{s+1}-x^s \\ u^{s+1}-u^s \end{matrix} \right|\right|^2\right] \end{aligned} \tag{29}$$

In the context of the Newton methods, in order to compute u^{s+1} that solves (13) to first order, the terms of order 2 and higher are truncated and the left hand side of (29) is set to zero. Furthermore x^{s+1} is substituted from (3) and the notation of Section II is introduced, yielding:

$$\begin{aligned} 0 = y^{s+1} - y^* + C^{s+1}\,\Phi^s(\chi(T;x^s,u^s)-x^s) + \\ (C^{s+1}\Gamma^s)(u^{s+1}-u^s) \end{aligned} \tag{30}$$

Solving (30) for u^{s+1} the control law is obtained:

$$u^{s+1} = u^s + (C^{s+1}\Gamma^s)^{-1}[C^{s+1}\Phi^s(x^s-\chi^s) + (y^*-y^{s+1})] \tag{31}$$

The algorithm is called quasi-Newton, because although it looks similar to a Newton method, strictly speaking it is not and consecutively it is not supported by the Kantorovic Theorem (Kantorovic and Akilow[12]). The reason is that in the Newton method x^{s+1} is specified by the solution of an equation of the form (12), while in the above derivation it was <u>arbitrarily</u> selected equal to $\chi(T;x^s,u^s)$ to conform with the evolution of the systems states. The stability of the control law is characterized by Theorem 2.

For linear systems, it can be shown (1) that (31) becomes:

$$\begin{aligned} u^{s+1} = u^s + [C(\exp(AT)-I)A^{-1}B]^{-1} \\ \cdot [C\exp(AT)(x^s-\chi^s) + y^* - y^{s+1}] \end{aligned} \tag{32}$$

For linear systems (32) is an output deadbeat controller that drives the system output to y^* within one sampling interval. Its properties are well studied (Kuo[13], Franklin and Powell[14]) and are not repeated here.

2. Second Quasi-Newton Control Law

An alternate way to derive a quasi-Newton control law is by considering the variation to first order of the output map around a state x^{s+1}. Then

$$\begin{aligned} g(x^{s+2}) - y^* = g(x^{s+1}) - y^* \\ + \partial g(\zeta)/\partial\chi\Big|_{\zeta=x^{s+1}} (x^{s+2}-x^{s+1}) + 0(||x^{s+2}-x^{s+1}||^2) \end{aligned} \tag{33}$$

x^{s+2} is the system state at $t = t^{s+2}$, i.e. $x^{s+2} = \chi(T;x^{s+1}, u^{s+1})$, a nonlinear function of u^{s+1}. If u^{s+1} is desired which makes the right hand side of (33) zero to first order (and produces $y^{s+2}=y^*$), the equation to be solved (after introducing the usual notation) is:

$$0 = y^{s+1} - y^* + C^{s+1}(\chi(T;x^{s+1},u^{s+1})-x^{s+1}) \tag{34}$$

The nonlinear equation (34) can be solved either by some iterative method (which is to be avoided in lieu of on-line calculations), or its solution can be approximated by the solution of an appropriate linear problem. This linear problem is obtained if $\chi(T;x^{s+1},u^{s+1})$ is approximated to first order by expansion around u^s:

$$\begin{aligned} \chi(T;x^{s+1},u^{s+1}) = \chi(T;x^{s+1},u^s) + \\ \frac{\partial\chi(T;x^{s+1},u^s)}{u^s}(u^{s+1}-u^s) + 0(||u^{s+1}-y^s||^2) \end{aligned} \tag{35}$$

The resulting control law (after some algebraic manipulation) is:

$$\begin{aligned} u^{s+1} = u^s + [C^{s+1}\hat{\Gamma}^s]^{-1}C(\chi^s-\hat{\chi}^s) \\ + [C^{s+1}\hat{\Gamma}^s]^{-1}(y^*-y^{s+1}) \end{aligned} \tag{36}$$

where the new quantities $\hat{x}^{s+2}$ and $\hat{\Gamma}^{s+1}$ have the following interpretation: $\hat{x}^{s+2} \stackrel{\text{def}}{=} \chi(T;x^{s+1},u^s)$ is the predicted system output at $t = t^{s+1}$ if the system input is to be held at u^s over the $(s+1)^{th}$ sampling interval. $\hat{\Gamma}^{s+1} \stackrel{\text{def}}{=} \partial x^{s+1}/\partial u^s$ is the corresponding derivative, obtained as in Section II.

(36), similar to (31) is not a formal Newton algorithm. Its stability is characterized by Theorem 2. For linear systems, the resulting control law is

$$\begin{aligned} u^{s+1} = u^s + [C(\exp(AT)-I)A^{-1}B]^{-1} \\ \cdot [C\exp(AT)(x^s-\chi^s) + y^* - y^{s+1}] \end{aligned} \tag{37}$$

This shows that for linear systems (37) and (32) are identical, therefore (37) is also an output deadbeat controller.

Asymptotic behavior for large sampling times

It can be shown that as the sampling time $T \to \infty$, (31) and (36) become identical to a basic Newton method for the solution of the system of algebraic nonlinear equations that characterize the steady state of the system governed by (1), i.e. computation of u^* such that

$$y^* = g(x(u^*))$$

where $x(u)$ is given by the implicit function

$$f(x,u) = 0$$

The implication is that the control law stability for large sampling times can be studied in terms of the convergence properties of a Newton algorithm for the solution of a system of algebraic equations.

Modified Newton Algorithms

When the s^{th} iterate of the Newton method is relatively far away from the solution, it is common practice to improve the convergence properties by introducing the relaxation factor λ. To every control law (31), (36), and (32) (or (37)) corresponds a modified algorithm, derived by relaxing the updates by λ:

$$u^{s+1} = u^s + \lambda(C^{s+1}\Gamma^s)^{-1}[C^{s+1}\Phi^s(x^s-x^s) + (y^*-y^{s+1})] \qquad (31')$$

$$u^{s+1} = u^s + \lambda[C^{s+1}\hat{\Gamma}^s]^{-1}C(\chi^s-\hat{\chi}^s) + \lambda[C^{s+1}\hat{\Gamma}^s]^{-1}(y^*-y^{s+1}) \qquad (36')$$

$$u^{s+1} = u^s + \lambda[C(\exp(AT)-I)A^{-1}B]^{-1} \cdot [C\exp(AT)(x^s-\chi^s) + y^* - y^{s+1}] \qquad (32')$$

λ can be used as an on-line tuning parameter for the respective control laws. An additional tuning parameter is the number n of forward steps.

VI. EXAMPLES

Example 1. This example shows the implications of the developed theory for linear controller design through the quasi-Newton method. Consider the system

$$\frac{\partial x_1}{\partial t} = -2x_1 + x_2$$

$$\frac{\partial x_2}{\partial t} = x_1 - 2x_2 + u \qquad (39)$$

$$y = x_2$$

Using a sampling time of 0.5 and the quasi-Newton linear control law (32), Fig. 1 shows the system response to a unit step disturbance at the output, where the dead-beat action is apparent.

Example 2. The following differential-algebraic equations typically appear in modelling Continuous Stirred Tank Reactors (CSTR) with reversible reactions. They are derived from differential mass and energy balances:

$$\frac{\partial x_1}{\partial t} = \frac{1}{\tau}(A_i-x_1) - k_A e^{Q_A/Rx_3} + k_R e^{-Q_R/Rx_3}$$

$$\frac{\partial x_2}{\partial t} = \frac{1}{\tau}(R_i-x_2) + k_A e^{-Q_A/Rx_3} - k_R e^{-Q_R/Rx_3}$$

$$\frac{\partial x_3}{\partial t} = \frac{1}{\tau}(T_i-x_3) + \frac{-\Delta H}{\rho c_p}(k_A e^{-Q_A/Rx_3} - k_R e^{-Q_R/Rx_3}$$

$$y = x_2$$

where x_1 and x_2 are reactant concentrations, x_3 is reactor temperature and T_i is feed stream temperature, the control input, the other variables being reaction constants. In Fig. 2 the steady state equilibrium diagram (solid line) and the d.c. gain of the linearized system (dashed line) are shown for a particular set of the parameters. It is noted that the system gain changes drastically in the operating region, at a specific input temperature it even changes sign.

The Quasi Newton control law (36) is used to design a nonlinear controller for the reactor. A linear controller design (IMC, Garcia and Morari[6]) is employed for comparison.

The control objective is to operate the reactor as close as possible to the maximum conversion point. Two runs of the reactor are presented. In the first case, the system has drifted to a point downhill on the left of maximum conversion, while in the second case the system has drifted downhill to the right. The goal of the respective control schemes is to recover the system in either case.

Figure 3 shows that both controllers perform equally well in the first case. Fig. 4 however shows that while the quasi-Newton controller displays approximately the same behavior in the second case, the linear IMC controller is unstable.

It should be noted that the inadequacy of the linear controller is not due to the particular design. Every linear controller with integral action will display similar behavior (Morari[15]). Removal of integral action control will result in large offsets as a result of gain variations. At the same time, since the steady state gain of the system changes sign, no adaptation mechanism could perform satisfactorily in this case either.

VII. CONCLUSIONS

The basic concept presented is that for a number of meaningful objectives, the control problem can be formulated as an operator inversion problem. Controller synthesis then is based on iterative solution algorithm development and stability analysis is based on algorithm convergence properties. Controller designs for nonlinear systems, induced from the Newton method were developed as an illustration of potential applications and their properties were investigated in the light of established results from operator theory. Examples were worked out for the case of a chemical reactor with rich nonlinear characteristics. The concept lends itself to extensions, which include gradient optimization induced controllers, distributed parameter systems, systems of mixed differential and algebraic equations and finally stabilization of unstable nonlinear systems.

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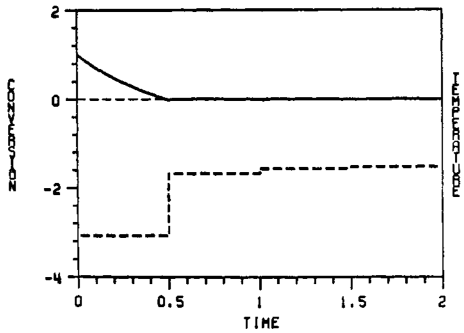

Figure 1. Quasi-Newton control for system (—) system output, (---) control inputs.

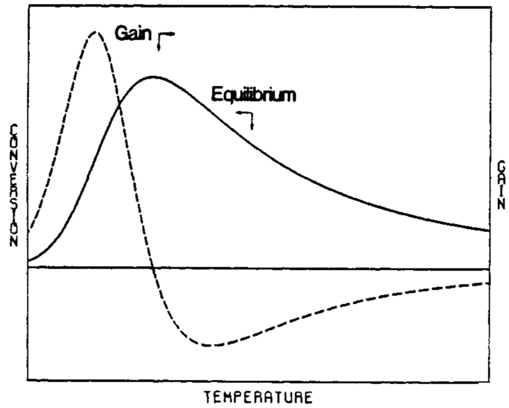

Figure 2. CSTR steady state temperature-conversion plane system (39). (—) system output.

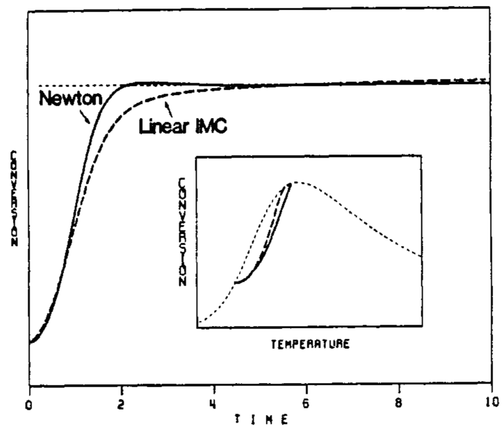

Figure 3. Quasi-Newton control for CSTR ( ) system output, (---) setpoint Inset is the reactor trajectory in the temperature-conversion plane (—), and the Newton, (*) IMC.

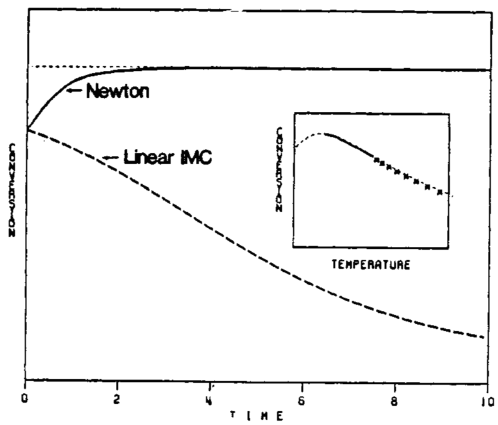

Figure 4. Quasi-Newton vs. linear IMC control for CSTR (—) system output, (---) setpoint